AN ELEMENTARY PROOF OF STIRLING'S FORMULA

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1. Introduction. Stirling's formula is

$$\Gamma(\alpha) \approx \left(\frac{\alpha-1}{e}\right)^{\alpha-1}\sqrt{2\pi(\alpha-1)} \tag{1}$$

as $\alpha \to \infty$, in the sense that the ratio of the two sides tends to 1. By definition,

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}\,dx \quad \text{for } \alpha > 0. \tag{2}$$

Thus, for positive integers n, $\Gamma(n) = (n-1)!$.

The object of this expository note is to give a short but complete version of Laplace's argument for (1). This will be done in the next section, but here is the idea. Write $\exp(x) = e^x$ and $\beta = \alpha - 1$. Then

$$\Gamma(\alpha) = \int_0^\infty \exp\{\Psi_\beta(x)\}\,dx, \tag{3}$$

where

$$\Psi_\beta(x) = \beta\log x - x. \tag{4}$$

Now Ψ_β has its maximum at $x = \beta$, and

$$\Psi_\beta(\beta + y) \doteq \beta\log\beta - \beta - \frac{1}{2\beta}y^2. \tag{5}$$

Here, $\doteq$ means "nearly equal," and is used informally. So

$$\begin{aligned}\Gamma(\alpha) &\doteq \exp(\beta\log\beta - \beta)\cdot\int_{-\infty}^\infty \exp\left(-\frac{1}{2\beta}y^2\right)dy \\ &= \left(\frac{\beta}{e}\right)^\beta\sqrt{2\pi\beta},\end{aligned} \tag{6}$$

because

$$\int_{-\infty}^\infty \exp\left(-\frac{1}{2}z^2\right)dz = \sqrt{2\pi}. \tag{7}$$

2. The argument. Recall Ψ_β from (4). The rigorous version of (5) is the following identity:

$$\Psi_\beta(\beta + y) = \beta\log\beta - \beta - \beta g(y/\beta), \tag{8}$$

where

$$g(v) = v - \log(1 + v). \tag{9}$$

Substitute (8) into (3) and change variables:

$$\Gamma(\alpha) = \left(\frac{\beta}{e}\right)^\beta\int_{-\beta}^\infty \exp\{-\beta g(y/\beta)\}\,dy. \tag{10}$$

Change variables again, putting $y = \sqrt{\beta}\,z$, getting

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(11) $$\Gamma(\alpha) = \left(\frac{\beta}{e}\right)^{\beta} \sqrt{2\pi\beta}\, \Gamma_1(\beta),$$

where

(12) $$\Gamma_1(\beta) = \frac{1}{\sqrt{2\pi}} \int_{-\sqrt{\beta}}^{\infty} \exp\{-\beta g(z/\sqrt{\beta})\}\, dz.$$

Our proof of Stirling's formula is reduced to the following.

LEMMA.

$$\lim_{\beta \to \infty} \Gamma_1(\beta) = 1.$$

Proof. Fix L large but finite. Then $\Gamma_1(\beta) = \Gamma_L(\beta) + \tau_L(\beta)$, where

(13) $$\Gamma_L(\beta) = \frac{1}{\sqrt{2\pi}} \int_{-L}^{L} \exp\{-\beta g(z/\sqrt{\beta})\}\, dz$$
$$\to \frac{1}{\sqrt{2\pi}} \int_{-L}^{L} \exp\left\{-\frac{1}{2}z^2\right\} dz \quad \text{as } \beta \to \infty,$$

because

(14) $$\beta g(z/\sqrt{\beta}) \to \frac{1}{2}z^2 \quad \text{as } \beta \to \infty, \text{ uniformly for } z \in [-L, L].$$

Relationship (14) holds because for v small enough,

(15) $$(1-\varepsilon)\frac{1}{2}v^2 < g(v) < (1+\varepsilon)\frac{1}{2}v^2.$$

It remains to estimate $\tau_L(\beta)$. We take the upper tail first. Abbreviate $h(z) = \beta g(z/\sqrt{\beta})$; the dependence of h on β is implicit. Then

(16) $$\int_L^{\infty} \exp\{-h(z)\}\, dz \leqslant \frac{1}{h'(L)} \int_L^{\infty} h'(z) \exp\{-h(z)\}\, dz$$
$$= \frac{1}{h'(L)} \exp\{-h(L)\}$$
$$\to \frac{1}{L} \exp\left\{-\frac{1}{2}L^2\right\} \quad \text{as } \beta \to \infty;$$

the first inequality holds because $h'(z) = \sqrt{\beta}\, z/(\sqrt{\beta} + z)$ is increasing in $z > 0$. The lower tail is similar, so for β large,

(17) $$\tau_L(\beta) \leqslant \frac{2}{\sqrt{2\pi}} \frac{1}{L} \exp\left\{-\frac{1}{2}L^2\right\} + \varepsilon,$$

which completes the proof.

The present approximation comes out in powers of $\beta = \alpha - 1$; if powers of α are preferred, a preliminary change of variables can be made ($x = e^u$), so

(18) $$\Gamma(\alpha) = \int_{-\infty}^{\infty} u^{\alpha} \exp(-e^u)\, du.$$

A refinement of the argument leads to the usual asymptotic development,

(19) $$\Gamma_1(\beta) = 1 + \frac{1}{12\beta} + \cdots.$$

3. History. The first proofs of Stirling's formula were given by de Moivre (1730) and Stirling

(1730). Both used what is now called the Euler–MacLaurin formula to approximate $\log 2 + \log 3 + \cdots + \log n$. De Moivre proved the result on the way to the normal approximation for the binomial distribution. His first derivation did not explicitly determine the constant $\sqrt{2\pi}$. In a 1731 addendum, he acknowledged that Stirling was able to determine the constant, using Wallis' formula. To statisticians, the most familiar version of this argument is Robbins (1955); also see Feller (1968). The approach yields upper and lower bounds for $n!$ but does not extend to $\Gamma(\alpha)$.

In essence, we are using a direct form of Laplace's (1774) method to estimate the integral in (12), with quite explicit bounds. For a more general treatment, see de Bruijn (1981, sect. 4.5). For one very similar to ours, see Woodroofe (1975, p. 127). We know of three other approaches to proving Stirling's formula. Modern analysts extend Γ into the complex plane, and have a proof of (1) using the saddlepoint method: see de Bruijn (1981, sect. 6.9). Artin (1964) presents a fascinating discussion of the Γ-function and its properties, as well as a proof of Stirling's formula based on the following theorem: *$\Gamma(\alpha)$ is the only log convex function on $(0, \infty)$ satisfying $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$, with $\Gamma(1) = 1$.* A third approach using the residue calculus is due to Lindelof; for a modern exposition, see Ahlfors (1979).

We stumbled on our proof while working on finite forms of de Finetti's theorem for exponential families (Diaconis and Freedman, 1980, 1984). As an example, we were thinking of the gamma shape parameter $\alpha > 0$ in the family

$$\frac{1}{\Gamma(\alpha)} x^{\alpha-1} e^{-x} \quad \text{for } 0 \leqslant x < \infty. \tag{20}$$

As $\alpha \to \infty$, the density (20) tends to normal, with mean β and variance β. The argument for Stirling's formula was a by-product.

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A BOUND FOR THE NUMBER OF MULTIPLICATIVE PARTITIONS

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Two factorizations of a positive integer are considered to be essentially the same if they differ only in the order of the factors. For example, the four essentially different factorizations of 12 are 12, $6 \cdot 2$, $4 \cdot 3$, and $2 \cdot 2 \cdot 3$. For a positive integer n, let $f(n)$ be the number of essentially different factorizations of n. In a recent note [1] in this MONTHLY, J. F. Hughes and J. O. Shallit proved that $f(n) \leqslant 2n^{\sqrt{2}}$ and conjectured that $f(n) \leqslant n$. We show that this conjecture can be settled in the affirmative in a manner much simpler than they expected.