



An Elementary Proof of Stirling's Formula

P. Diaconis; D. Freedman

The American Mathematical Monthly, Vol. 93, No. 2. (Feb., 1986), pp. 123-125.

Stable URL:

<http://links.jstor.org/sici?sici=0002-9890%28198602%2993%3A2%3C123%3AAEPOSF%3E2.0.CO%3B2-7>

The American Mathematical Monthly is currently published by Mathematical Association of America.

Your use of the JSTOR archive indicates your acceptance of JSTOR's Terms and Conditions of Use, available at <http://www.jstor.org/about/terms.html>. JSTOR's Terms and Conditions of Use provides, in part, that unless you have obtained prior permission, you may not download an entire issue of a journal or multiple copies of articles, and you may use content in the JSTOR archive only for your personal, non-commercial use.

Please contact the publisher regarding any further use of this work. Publisher contact information may be obtained at <http://www.jstor.org/journals/maa.html>.

Each copy of any part of a JSTOR transmission must contain the same copyright notice that appears on the screen or printed page of such transmission.

The JSTOR Archive is a trusted digital repository providing for long-term preservation and access to leading academic journals and scholarly literature from around the world. The Archive is supported by libraries, scholarly societies, publishers, and foundations. It is an initiative of JSTOR, a not-for-profit organization with a mission to help the scholarly community take advantage of advances in technology. For more information regarding JSTOR, please contact support@jstor.org.

AN ELEMENTARY PROOF OF STIRLING'S FORMULA

P. DIACONIS¹*Statistics Department, Stanford University, Stanford, CA 94305*D. FREEDMAN²*Statistics Department, University of California, Berkeley, CA 94720***1. Introduction.** Stirling's formula is

$$(1) \quad \Gamma(\alpha) \approx \left(\frac{\alpha-1}{e}\right)^{\alpha-1} \sqrt{2\pi(\alpha-1)}$$

as $\alpha \rightarrow \infty$, in the sense that the ratio of the two sides tends to 1. By definition,

$$(2) \quad \Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx \quad \text{for } \alpha > 0.$$

Thus, for positive integers n , $\Gamma(n) = (n-1)!$.

The object of this expository note is to give a short but complete version of Laplace's argument for (1). This will be done in the next section, but here is the idea. Write $\exp(x) = e^x$ and $\beta = \alpha - 1$. Then

$$(3) \quad \Gamma(\alpha) = \int_0^\infty \exp\{\Psi_\beta(x)\} dx,$$

where

$$(4) \quad \Psi_\beta(x) = \beta \log x - x.$$

Now Ψ_β has its maximum at $x = \beta$, and

$$(5) \quad \Psi_\beta(\beta + y) \doteq \beta \log \beta - \beta - \frac{1}{2\beta} y^2.$$

Here, \doteq means "nearly equal," and is used informally. So

$$(6) \quad \begin{aligned} \Gamma(\alpha) &\doteq \exp(\beta \log \beta - \beta) \cdot \int_{-\infty}^\infty \exp\left(-\frac{1}{2\beta} y^2\right) dy \\ &= \left(\frac{\beta}{e}\right)^\beta \sqrt{2\pi\beta}, \end{aligned}$$

because

$$(7) \quad \int_{-\infty}^\infty \exp\left(-\frac{1}{2} z^2\right) dz = \sqrt{2\pi}.$$

2. The argument. Recall Ψ_β from (4). The rigorous version of (5) is the following identity:

$$(8) \quad \Psi_\beta(\beta + y) = \beta \log \beta - \beta - \beta g(y/\beta),$$

where

$$(9) \quad g(v) = v - \log(1 + v).$$

Substitute (8) into (3) and change variables:

$$(10) \quad \Gamma(\alpha) = \left(\frac{\beta}{e}\right)^\beta \int_{-\beta}^\infty \exp\{-\beta g(y/\beta)\} dy.$$

Change variables again, putting $y = \sqrt{\beta} z$, getting¹Research partially supported by National Science Foundation Grant MCS80-24649.²Research partially supported by National Science Foundation Grant MCS83-01812.

$$(11) \quad \Gamma(\alpha) = \left(\frac{\beta}{e}\right)^\beta \sqrt{2\pi\beta} \Gamma_1(\beta),$$

where

$$(12) \quad \Gamma_1(\beta) = \frac{1}{\sqrt{2\pi}} \int_{-\sqrt{\beta}}^{\infty} \exp\{-\beta g(z/\sqrt{\beta})\} dz.$$

Our proof of Stirling's formula is reduced to the following.

LEMMA.

$$\lim_{\beta \rightarrow \infty} \Gamma_1(\beta) = 1.$$

Proof. Fix L large but finite. Then $\Gamma_1(\beta) = \Gamma_L(\beta) + \tau_L(\beta)$, where

$$(13) \quad \begin{aligned} \Gamma_L(\beta) &= \frac{1}{\sqrt{2\pi}} \int_{-L}^L \exp\{-\beta g(z/\sqrt{\beta})\} dz \\ &\rightarrow \frac{1}{\sqrt{2\pi}} \int_{-L}^L \exp\left\{-\frac{1}{2}z^2\right\} dz \quad \text{as } \beta \rightarrow \infty, \end{aligned}$$

because

$$(14) \quad \beta g(z/\sqrt{\beta}) \rightarrow \frac{1}{2}z^2 \quad \text{as } \beta \rightarrow \infty, \text{ uniformly for } z \in [-L, L].$$

Relationship (14) holds because for v small enough,

$$(15) \quad (1 - \varepsilon) \frac{1}{2}v^2 < g(v) < (1 + \varepsilon) \frac{1}{2}v^2.$$

It remains to estimate $\tau_L(\beta)$. We take the upper tail first. Abbreviate $h(z) = \beta g(z/\sqrt{\beta})$; the dependence of h on β is implicit. Then

$$(16) \quad \begin{aligned} \int_L^{\infty} \exp\{-h(z)\} dz &\leq \frac{1}{h'(L)} \int_L^{\infty} h'(z) \exp\{-h(z)\} dz \\ &= \frac{1}{h'(L)} \exp\{-h(L)\} \\ &\rightarrow \frac{1}{L} \exp\left\{-\frac{1}{2}L^2\right\} \quad \text{as } \beta \rightarrow \infty; \end{aligned}$$

the first inequality holds because $h'(z) = \sqrt{\beta} z/(\sqrt{\beta} + z)$ is increasing in $z > 0$. The lower tail is similar, so for β large,

$$(17) \quad \tau_L(\beta) \leq \frac{2}{\sqrt{2\pi}} \frac{1}{L} \exp\left\{-\frac{1}{2}L^2\right\} + \varepsilon,$$

which completes the proof.

The present approximation comes out in powers of $\beta = \alpha^{-1}$; if powers of α are preferred, a preliminary change of variables can be made ($x = e^u$), so

$$(18) \quad \Gamma(\alpha) = \int_{-\infty}^{\infty} u^\alpha \exp(-e^u) du.$$

A refinement of the argument leads to the usual asymptotic development,

$$(19) \quad \Gamma_1(\beta) = 1 + \frac{1}{12\beta} + \dots$$

3. History. The first proofs of Stirling's formula were given by de Moivre (1730) and Stirling

(1730). Both used what is now called the Euler–MacLaurin formula to approximate $\log 2 + \log 3 + \cdots + \log n$. De Moivre proved the result on the way to the normal approximation for the binomial distribution. His first derivation did not explicitly determine the constant $\sqrt{2\pi}$. In a 1731 addendum, he acknowledged that Stirling was able to determine the constant, using Wallis' formula. To statisticians, the most familiar version of this argument is Robbins (1955); also see Feller (1968). The approach yields upper and lower bounds for $n!$ but does not extend to $\Gamma(\alpha)$.

In essence, we are using a direct form of Laplace's (1774) method to estimate the integral in (12), with quite explicit bounds. For a more general treatment, see de Bruijn (1981, sect. 4.5). For one very similar to ours, see Woodroffe (1975, p. 127). We know of three other approaches to proving Stirling's formula. Modern analysts extend Γ into the complex plane, and have a proof of (1) using the saddlepoint method: see de Bruijn (1981, sect. 6.9). Artin (1964) presents a fascinating discussion of the Γ -function and its properties, as well as a proof of Stirling's formula based on the following theorem: $\Gamma(\alpha)$ is the only log convex function on $(0, \infty)$ satisfying $\Gamma(\alpha + 1) = \alpha\Gamma(\alpha)$, with $\Gamma(1) = 1$. A third approach using the residue calculus is due to Lindelof; for a modern exposition, see Ahlfors (1979).

We stumbled on our proof while working on finite forms of de Finetti's theorem for exponential families (Diaconis and Freedman, 1980, 1984). As an example, we were thinking of the gamma shape parameter $\alpha > 0$ in the family

$$(20) \quad \frac{1}{\Gamma(\alpha)} x^{\alpha-1} e^{-x} \quad \text{for } 0 \leq x < \infty.$$

As $\alpha \rightarrow \infty$, the density (20) tends to normal, with mean β and variance β . The argument for Stirling's formula was a by-product.

References

1. L. Ahlfors, *Complex Analysis*, 3rd ed., McGraw-Hill, New York, 1979.
2. E. Artin, *The Gamma Function*, Holt, Rinehart, Winston, New York, 1964.
3. N. G. de Bruijn, *Asymptotic Methods in Analysis*, Dover, New York, 1981.
4. A. de Moivre, *Miscellanea Analytica de Seriebus et Quadraturis*, London, 1730.
5. P. Diaconis and D. Freedman, Finite exchangeable sequences, *Ann. Probab.*, 8 (1980) 745–764.
6. _____, Finite versions of de Finetti's theorem for certain exponential families, Technical report, University of California, Berkeley, 1984.
7. W. Feller, *An Introduction to Probability Theory and Its Application*, Vol. 1, 3rd ed., Wiley, New York, 1968.
8. P. S. Laplace, Mémoire sur la probabilité des causes par les événements. *Mémoires de mathématique et de Physique Présentés à l'Académie Royale des Sciences, par Divers Savants, & Lus dans ses Assemblées*, 6, 1774 (Reprinted in Laplace's *Oeuvres Complètes*, 8, 27–65, English translation by S. Stigler, Technical Report #164, Department of Statistics, University of Chicago, September, 1984.)
9. H. Robbins, A remark on Stirling's formula, this MONTHLY, 62 (1955) 26–29.
10. J. Stirling, *Methodus Differentialis*, London, 1730.
11. M. Woodroffe, *Probability with applications*, McGraw-Hill, New York, 1975.

A BOUND FOR THE NUMBER OF MULTIPLICATIVE PARTITIONS

L. E. MATTICS AND F. W. DODD

Department of Mathematics, University of South Alabama, Mobile, AL 36688

Two factorizations of a positive integer are considered to be essentially the same if they differ only in the order of the factors. For example, the four essentially different factorizations of 12 are 12, $6 \cdot 2$, $4 \cdot 3$, and $2 \cdot 2 \cdot 3$. For a positive integer n , let $f(n)$ be the number of essentially different factorizations of n . In a recent note [1] in this MONTHLY, J. F. Hughes and J. O. Shallit proved that $f(n) \leq 2n^{\sqrt{2}}$ and conjectured that $f(n) \leq n$. We show that this conjecture can be settled in the affirmative in a manner much simpler than they expected.